COMPLEX NUMBER

1. The complex number system

$z = a + ib$, then $a - ib$ is called congugate of z and is denoted by $\bar{z}$.

2. Equality In Complex Number:

$$z_1 = z_2 \quad \Rightarrow \quad Re(z_1) = Re(z_2) \text{ and } I_m(z_1) = I_m(z_2).$$

3. Properties of arguments

(i) $\arg(z_1z_2) = \arg(z_1) + \arg(z_2) + 2m\pi$ for some integer m.

(ii) $\arg(z_1/z_2) = \arg(z_1) - \arg(z_2) + 2m\pi$ for some integer m.

(iii) $\arg(z^2) = 2\arg(z) + 2m\pi$ for some integer m.

(iv) $\arg(z) = 0 \quad \Leftrightarrow \quad$ z is a positive real number

(v) $\arg(z) = \pm \pi/2 \quad \Leftrightarrow \quad$ z is purely imaginary and $z \neq 0$

4. Properties of conjugate

(i) $|z| = |\bar{z}|$ (ii) $z\bar{z} = |z|^2$ (iii) $\overline{z_1 + z_2} = \bar{z}_1 + \bar{z}_2$

(iv) $\overline{z_1 - z_2} = \bar{z}_1 - \bar{z}_2$ (v) $\overline{z_1z_2} = \bar{z}_1\,\bar{z}_2$

(vi) $\overline{\left(\dfrac{z_1}{z_2}\right)} = \dfrac{\bar{z}_1}{\bar{z}_2}$ $(z_2 \neq 0)$

(vii) $|z_1 + z_2|^2 = (z_1 + z_2)\overline{(z_1 + z_2)} = |z_1|^2 + |z_2|^2 + z_1\bar{z}_2 + \bar{z}_1 z_2$

(viii) $\overline{(\bar{z}_1)} = z$ (ix) If $w = f(z)$, then $\overline{w} = f(\bar{z})$

(x) $\arg(z) + \arg(\bar{z})$

5. Rotation theorem

If $P(z_1)$, $Q(z_2)$ and $R(z_3)$ are three complex numbers and $\angle PQR = \theta$, then

$$\left(\frac{z_3 - z_2}{z_1 - z_2}\right) = \left|\frac{z_3 - z_2}{z_1 - z_2}\right| e^{i\theta}$$

6. Demoivre's Theorem :

If n is any integer then

(i) $(\cos\theta + i\sin\theta)^n = \cos n\theta + i\sin n\theta$

(ii) $(\cos\theta_1 + i\sin\theta_1)(\cos\theta_2 + i\sin\theta_2)(\cos\theta_3 + i\sin\theta_2)(\cos\theta_3 + i\sin\theta_3) .....(\cos\theta_n + i\sin\theta_n) = \cos(\theta_1 + \theta_2 + \theta_3 + ... \theta_n) + i\sin(\theta_1 + \theta_2 + \theta_3 + ... + \theta_n)$

7. Cube Root Of Unity :

(i) The cube roots of unity are 1, $\frac{-1+i\sqrt{3}}{2}$, $\frac{-1-i\sqrt{3}}{2}$.

(ii) If ω is one of the imaginary cube roots of unity then $1 + \omega + \omega^2 = 0$. In general $1 + \omega^r + \omega^{2r} = 0$; where $r \in I$ but is not the multiple of 3.

8. Geometrical Properties:

Distance formula : $|z_1 - z_2|$.

Section formula : $z = \frac{mz_2 + nz_1}{m+n}$ (internal division), $z = \frac{mz_2 - nz_1}{m-n}$ (external division)

(1) $amp(z) = \theta$ is a ray emanating from the origin inclined at an angle θ to the x– axis.

(2) $|z - a| = |z - b|$ is the perpendicular bisector of the line joining a to b.

(3) If $\left|\frac{z - z_1}{z - z_2}\right| = k \neq 1, 0$, then locus of z is circle.

VECTORS

I. Position Vector Of A Point:

let O be a fixed origin, then the position vector of a point P is the vector $\overrightarrow{OP}$. If $\vec{a}$ and $\vec{b}$ are position vectors of two points A and B, then,

$\overrightarrow{AB} = \vec{b} - \vec{a}$ = pv of B – pv of A.

DISTANCE FORMULA : Distance between the two points A $(\vec{a})$ and B $(\vec{b})$ is AB = $|\vec{a} - \vec{b}|$

SECTION FORMULA : $\vec{r} = \frac{n\vec{a} + m\vec{b}}{m+n}$. Mid point of AB = $\frac{\vec{a} + \vec{b}}{2}$.

II. Scalar Product Of Two Vectors: $\vec{a}\,.\,\vec{b} = |\vec{a}|\,|\vec{b}|\cos\theta$, where $|\vec{a}|, |\vec{b}|$ are magnitude of $\vec{a}$ and $\vec{b}$ respectively and θ is angle between $\vec{a}$ and $\vec{b}$.

1. i.i = j.j = k.k = 1; i.j = j.k = k.i = 0 ☞ projection of $\vec{a}$ on $\vec{b} = \dfrac{\vec{a}.\vec{b}}{|\vec{b}|}$

2. If $\vec{a} = a_1 i + a_2 j + a_3 k$ & $\vec{b} = b_1 i + b_2 j + b_3 k$ then $\vec{a}.\vec{b} = a_1 b_1 + a_2 b_2 + a_3 b_3$

3. The angle ϕ between $\vec{a}$&$\vec{b}$ is given by $\cos\phi = \dfrac{\vec{a}.\vec{b}}{|\vec{a}|\,|\vec{b}|}$, $0 \le \phi \le \pi$

4. $\vec{a}\,.\,\vec{b} = 0 \Leftrightarrow \vec{a} \perp \vec{b}$ $(\vec{a} \ne 0\ \ \vec{b} \ne 0)$

III. Vector Product Of Two Vectors:

1. If $\vec{a}$&$\vec{b}$ are two vectors & θ is the angle between them then $\vec{a} x \vec{b} = |\vec{a}||\vec{b}|\sin\theta\vec{n}$, where $\vec{n}$ is the unit vector perpendicular to both $\vec{a}$&$\vec{b}$ such that $\vec{a}, \vec{b}$&$\vec{n}$ forms a right handed screw system.

2. Geometrically $|\vec{a} x \vec{b}|$ = area of the parallelogram whose two adjacent sides are represented by $\vec{a}$&$\vec{b}$.

3. $\hat{i}\times\hat{i} = \hat{j}\times\hat{j} = \hat{k}\times\hat{k} = \vec{0}$; $\hat{i}\times\hat{j} = \hat{k}, \hat{j}\times\hat{k} = \hat{i}, \hat{k}\times\hat{i} = \hat{j}$

4. If $\vec{a} = a_1\hat{i} + a_2\hat{j} + a_3\hat{k}$ & $\vec{b} = b_1\hat{i} + b_2\hat{j} + b_3\hat{k}$ then

$$\vec{a}\times\vec{b} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$$

5. $\vec{a}\times\vec{b} = \vec{o} \Leftrightarrow \vec{a}$ and $\vec{b}$ are parallel (collinear)

$(\vec{a} \ne 0, \vec{b} \ne 0)$ i.e. $\vec{a} = K\vec{b}$, where K is a scalar.

6. Unit vector perpendicular to the plane of $\vec{a}$ & $\vec{b}$ is $\hat{n} = \pm \dfrac{\vec{a} \times \vec{b}}{|\vec{a} \times \vec{b}|}$

☞ If $\vec{a}, \vec{b}$ & $\vec{c}$ are the pv's of 3 points A, B & C then the vector area of triangle

ABC = $\frac{1}{2}\left[\vec{a} \times \vec{b} + \vec{b} \times \vec{c} + \vec{c} \times \vec{a}\right]$.

The points A, B & C are collinear if $\vec{a} \times \vec{b} + \vec{b} \times \vec{c} + \vec{c} \times \vec{a} = \vec{0}$

☞ Area of any quadrilateral whose diagonal vectors are $\vec{d}_1$ & $\vec{d}_2$ is given by $\frac{1}{2}\left|\vec{d}_1 \times \vec{d}_2\right|$

☞ Lagrange's Identity : $(\vec{a} \times \vec{b})^2 = |\vec{a}|^2 |\vec{b}|^2 - (\vec{a}.\vec{b})^2 = \begin{vmatrix} \vec{a}.\vec{a} & \vec{a}.\vec{b} \\ \vec{a}.\vec{b} & \vec{b}.\vec{b} \end{vmatrix}$

IV. Scalar Triple Product:

☞ The scalar triple product of three vectors $\vec{a}$, $\vec{b}$ & $\vec{c}$ is defined as:

$$\vec{a} \times \vec{b} . \vec{c} = |\vec{a}| |\vec{b}| |\vec{c}| \sin\theta \cos\phi .$$

☞ Volume of tetrahydron $V = [\vec{a}\vec{b}\vec{c}]$

☞ In a scalar triple product the position of dot & cross can be interchanged i.e.

$$\vec{a}.(\vec{b} \times \vec{c}) = (\vec{a} \times \vec{b}).\vec{c} \quad \text{OR} \quad [\vec{a}\ \vec{b}\ \vec{c}] = [\vec{b}\ \vec{c}\ \vec{a}] = [\vec{c}\ \vec{a}\ \vec{b}]$$

☞ $\vec{a}.(\vec{b} \times \vec{c}) = -\vec{a}.(\vec{c} \times \vec{b})$ i.e. $[\vec{a}\ \vec{b}\ \vec{c}] = -[\vec{a}\ \vec{c}\ \vec{b}]$

☞ If $\vec{a} = a_1i + a_2j + a_3k$; $\vec{b} = b_1i + b_2j + b_3k$ & $\vec{c} = c_1i + c_2j + c_3k$ then $[\vec{a}\vec{b}\vec{c}] = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$.

☞ If $\vec{a}$, $\vec{b}$, $\vec{c}$ are coplanar $\Leftrightarrow [\vec{a}\vec{b}\vec{c}] = 0$.

☞ Volume of tetrahedron OABC with O as origin & A($\vec{a}$), B($\vec{b}$) and C($\vec{c}$) be the vertices = $\left| \frac{1}{6}[\vec{a}\ \vec{b}\ \vec{c}] \right|$

☞ The positon vector of the centroid of a tetrahedron if the pv's of its vertices are $\vec{a}$, $\vec{b}$, $\vec{c}$ & $\vec{d}$ are given by $\frac{1}{4}[\vec{a}+\vec{b}+\vec{c}+\vec{d}]$.

V. Vector Triple Product:

$\vec{a} \times (\vec{b} \times \vec{c}) = (\vec{a} \cdot \vec{c})\vec{b} - (\vec{a} \cdot \vec{b})\vec{c}$, $(\vec{a} \times \vec{b}) \times \vec{c} = (\vec{a} \cdot \vec{c})\vec{b} - (\vec{b} \cdot \vec{c})\vec{a}$

☞ $(\vec{a} \times \vec{b}) \times \vec{c} \neq \vec{a} \times (\vec{b} \times \vec{c})$, in general